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# Weak Scattering of Sound Waves in Random Media That Have Arbitrary Power-Law Spectra

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# Weak Scattering of Sound Waves in Random Media That Have Arbitrary Power-Law Spectra

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# Abstract

Log-amplitude and phase variances of a weakly scattered acoustic signal are calculated for line-of-sight propagation through a random medium. The spectrum of the index-of-refraction fluctuations in the random medium is assumed to scale in proportion to the wavenumber raised to an arbitrary power in the limit of large wavenumbers (small spatial scales). Both scalar and vector contributions to the index of refraction are considered. Most of the calculated results reduce to those given by Tatarskii (1971) and Ostashev (1994) when the power law exponent is –5/3, which is the value characteristic of turbulence. However, the results do not exactly reduce to an equation given by Flatté et al (1979) for the log-amplitude variance in terms of strength and diffraction parameters. The equation from Flatté et al is shown to be an approximation, valid only when the spectral energy in the random medium is concentrated at a well-defined outer scale.

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# 1. Introduction

Sound waves that have propagated through the atmosphere exhibit random behavior. The cause for this behavior in most cases is atmospheric turbulence, although other random atmospheric phenomena such as internal waves can also play a role. The random behavior of the sound wave can significantly impact acoustical systems used for target detection, identification, and location. Hence a capability to estimate the statistics of sound waves is needed to quantify and predict the performance of Army acoustical systems operating in atmospheric environments.

This report concerns the calculation of the variances in the phase and the logarithm of the amplitude (log amplitude) of a received signal. These variances have been the subject of much research during the past several decades. Two contrasting formulations of the variance statistics can be found in the literature: one originating with the pioneering work on random scattering by Tatarskii (1971), and the other with the elegant treatment of propagation through the random ocean by Flatté et al (1979). Tatarskii's results are specifically for the scattering of waves by turbulent phenomena; Flatté et al adopt a more general approach, although its application to turbulence is not obvious from their book. More recently, Ostashev (1994) pointed out that the results from researchers such as Tatarskii and Flatté et al apply strictly to scattering driven by fluctuations in an isotropic scalar field, as opposed to fluctuations in an isotropic vector field. In the atmosphere, for example, this would mean that the previous results apply only to scattering by random temperature fluctuations, and not to scattering by wind velocity fluctuations. This is a significant concern, since sound scattering by atmospheric turbulence comes primarily from the velocity fluctuations. Fortunately, Ostashev (1994) was able to extend Tatarskii's results to isotropic vector fields.

This report bridges the approaches of Tatarskii and Flatté et al, and extends their results to velocity fluctuations. To this end, I devise a very general formulation of wave scattering by a random medium that has an arbitary power-law spectrum. The formulation is compared to the results of Tatarskii and Flatté et al. Specificially, I consider the scattering of plane waves by scalar and vector fields, and the scattering of spherical waves by scalar and vector fields.

The report presumes that the reader has some familiarity with the literature on propagation through random media, such as the books by

Flatté et al and Tatarskii, or Rytov et al (1989). The reader may also find my earlier work (Wilson, 1998b) a useful reference on modeling of turbulence spectra and correlations.

## 2. Review of Previous Formulations for Weak Scattering

Flatté et al (1979) analyzed the scattering of waves by random media using two parameters: a strength parameter $\Phi$ and a diffraction parameter $\Lambda$. These parameters are given by the equations

$$\Phi^2 = 2\sigma^2 \mathcal{L} k_0^2 X, \text{ and} \tag{1}$$

$$\Lambda = \frac{X}{6L^2 k_0}. \tag{2}$$

In the preceding equations, $\sigma^2$ is the variance of the (normalized) sound-speed fluctuations,* $\mathcal{L}$ is the integral length scale, $L$ is the correlation length scale of the sound-speed fluctuations, $k_0$ is the acoustic wavenumber ($= 2\pi f/c_0$, where $f$ is the frequency and $c_0$ the mean sound speed), and $X$ is the propagation distance. This formulation involves two distinct length scales: the integral and correlation length scales. The integral length scale is calculated by integrating the correlation function with respect to separation, and then dividing by the variance. As a result, $\mathcal{L}$ is representative of the size of the largest, most energetic motions in the flow. On the other hand, the correlation length scale is defined by an expansion of the correlation function for small arguments and is representative of the outer scale, where the power-law scaling becomes invalid. The correlation length scale will be defined more precisely in the next section.

Those situations where the strength parameter is much less than 1 are termed *weak scattering*. A large strength parameter indicates *strong scattering*. Similarly, *weak* and *strong diffraction* correspond to small and large values of the diffraction parameter, respectively. Only weak scattering is considered in this report, since the statistical properties of strong scattering are already well established (Flatté et al, 1979). (In strong scattering, the complex acoustic signal has independent real and imaginary parts possessing zero-mean Gaussian statistics.) Weak

*For an acoustic wave propagating through the atmosphere, the normalized sound-speed fluctuation is

$$N \simeq \frac{T'}{2T_0} + \frac{v'}{c_0},$$

in which $T'$ is the fluctuation in temperature about the mean temperature $T_0$, and $v'$ is the fluctuation in the wind speed (along the direction of propagation) about the mean.

scattering is in many ways the more difficult problem. More importantly, it is also pertinent to most Army scenarios, where the frequencies are low and the propagation distances are moderate.

For weak scattering and strong diffraction ($\Phi \ll 1, \Lambda \gg 1$), Flatté et al gave the results

$$\left\langle \phi^2 \right\rangle = \frac{\Phi^2}{2}, \left\langle \chi^2 \right\rangle = \frac{\Phi^2}{2}, \tag{3}$$

where $\langle \phi^2 \rangle$ and $\langle \chi^2 \rangle$ are the phase and log-amplitude variances, respectively. For weak scattering and weak diffraction ($\Phi \ll 1, \Lambda \ll 1$), the phase variance is

$$\left\langle \phi^2 \right\rangle = \Phi^2. \tag{4}$$

The calculation of the log-amplitude fluctuations for weak scattering and weak diffraction—the *geometrical acoustics* regime—is the most challenging problem. Flatté et al gave but did not present a derivation for the equation (their equation (8.3.2))

$$\left\langle \chi^2 \right\rangle \approx \frac{1}{4}\Lambda\Phi^2. \tag{5}$$

Following this equation, they state:

> *The log-amplitude fluctuations in the geometrical-optics region are thus of second order, and a careful treatment yields (8.3.2), where the exact coefficient of $\Lambda\Phi^2$ depends on the details of the ocean-fluctuation spectrum . . . but is of order unity.*

A primary purpose of this report is to explore the general validity of equation (5).

One issue concerning the general validity of equation (5) is evident in Flatté et al's remark that the coefficient (1/4 in the equation) depends on the "details of the ocean-fluctuation spectrum." And so, we ask the following: How much does the coefficient actually vary? On what properties of the propagation medium does it depend? Can the coefficient be defined more precisely?

A broader issue concerns whether the general relationship that $\langle \chi^2 \rangle$ is proportional to $\Lambda\Phi^2$ holds for propagation through random media besides the ocean. The main interest of Flatté et al was propagation through ocean internal (buoyancy) waves. Does their result apply to propagation through

turbulence? An important clue to answering this question is in the authors' discussion in section 6.2 of their book regarding the transition from the geometric acoustics to the partially saturated scattering regime. There they state that the boundary between these two regimes is given by $\Lambda^{p/2}\Phi^2 = 1$, where $p$ is the exponent in the power-law spectrum for the random fluctuations ($F(k) \propto k^{-p}$). Based on well-known arguments first made by Kolmogorov, $p = 5/3$ for turbulence. However, for the internal waves considered by Flatté et al, $p = 2$. Therefore, we might anticipate that

$$\left\langle \chi^2 \right\rangle \propto \Lambda^{p/2}\Phi^2. \tag{6}$$

As discussed earlier, equations for the log-amplitude variance have been derived for propagation through turbulence by Tatarskii (1971). For plane waves,

$$\left\langle \chi^2 \right\rangle = 0.308 C_N^2 k_0^{7/6} X^{11/6}, \tag{7}$$

and for spherical waves,

$$\left\langle \chi^2 \right\rangle = 0.124 C_N^2 k_0^{7/6} X^{11/6}, \tag{8}$$

in which $C_N^2$ is called the *structure-function parameter* for the index-of-refraction fluctuations. The results for plane and spherical waves differ only with regard to their numerical coefficients. If one substitutes the values for $\Phi$ and $\Lambda$ in equations (1) and (2) into equation (6), the dependence on frequency and propagation distance agrees with Tatarskii's results. Nonetheless, there is an indication that something is amiss: equation (6) involves two parameters associated with the large-scale (energy-subrange) structure of the turbulence, $\sigma^2$ and $\mathcal{L}$. In equation (7), $C_N^2$ is strictly a property of the small-scale (inertial-subrange) structure of the turbulence. Unless the product $\sigma^2\mathcal{L}$ in equation (1) has a universal relationship to $C_N^2$, there is a fundamental disagreement between the results. The solution to this quandry, as well as to the other questions posed earlier, will become apparent in the following sections.

## 3. Turbulence Modeling—Assumptions and Parameters

In this section, I discuss the statistical assumptions regarding the fluctuations in the propagation medium that will be used to derive the propagation statistics. Only two basic assumptions are made; additional assumptions are avoided so as not to restrict applicability of the results. The first assumption is simply that the medium has a finite variance and integral length scale. This leads to the requirement for the one-dimensional (1D) spectrum $F(k)$ of the fluctuations in the propagation medium that (Wilson, 1998b)

$$F(k=0) = \frac{\sigma^2 \mathcal{L}}{\pi}. \tag{9}$$

The second assumption is that for large wavenumber $k \gg \mathcal{L}$, the spectrum obeys a power-law scaling:

$$F(k) = \gamma k^{-p}, \tag{10}$$

where $\gamma$ is some constant (which may depend on the properties of the flow), and $p$ is the power-law exponent. Equivalently, we can set

$$F(k) = \frac{\sigma^2 \mathcal{L}}{\pi} S(k\mathcal{L}), \tag{11}$$

where $S(k\mathcal{L})$ is a function equal to 1 for $k = 0$, and proportional to $\beta (k\mathcal{L})^{-p}$ at large wavenumber, where

$$\beta = \frac{\pi \gamma \mathcal{L}^{p-1}}{\sigma^2}. \tag{12}$$

An alternative formulation to equation (10) involves the structure function. The structure function for separations parallel to the direction of propagation is by definition

$$D_{\parallel}(r) = \left\langle [N(r) - N(0)]^2 \right\rangle = C_N^2 r^{p-1}. \tag{13}$$

(When the preceding equation is applied to the velocity field, the separation $r$ between the measurement points is implicitly in the same

direction as the velocity component being measured.) A relationship between $\gamma$ and $C_N^2$ can be derived using the relationship

$$D_{\parallel}(r) = 4\int_0^{\infty} (1 - \cos kr)\, F(k)\, dk. \tag{14}$$

The function $(1 - \cos kr)$ filters out the dependence of $F(k)$ on small wavenumbers. Hence we can substitute equation (10) into (14). Integration then leads to

$$\frac{C_N^2}{\gamma} = -4\Gamma(1-p)\sin\frac{p\pi}{2}. \tag{15}$$

A second alternative formulation to equation (10) was used by Flatté et al. For the correlation function at small separations, they wrote

$$R_{\perp}(r) = \sigma^2\left(1 - \left|\frac{r}{L}\right|^{p-1}\right), \tag{16}$$

where the subscripted $\perp$ means that the separation between the measurement points is perpendicular to the direction of propagation. (In an isotropic scalar field, it does not matter in which direction the displacement is taken. For an isotropic vector field, however, the displacement direction is relevant.) For scalars, it follows directly from the definition of the structure function that

$$C_N^2 = \frac{2\sigma^2}{L^{p-1}}. \tag{17}$$

For a divergenceless vector field (such as velocity fluctuations in turbulence),

$$R_{\perp}(r) = R_{\parallel}(r) + \frac{r}{2}\frac{dR_{\parallel}(r)}{dr}. \tag{18}$$

Using this relationship, we derive for vector fields

$$C_N^2 = \frac{4\sigma^2}{(p+1)\,L^{p-1}}. \tag{19}$$

# 4. Derivation of the Log-Amplitude and Phase Variances

Under the conditions of weak scattering, Rytov's method can show that the variance of log amplitude and phase for a propagating plane wave are (Rytov et al, 1989)

$$\left\langle \chi^2, \phi^2 \right\rangle = 2\pi^2 k_0^2 X \int_0^\infty \left[ 1 \mp \frac{k_0}{X k_\perp^2} \sin \frac{X k_\perp^2}{k_0} \right] \Theta\left(0, k_\perp\right) k_\perp \, dk_\perp, \tag{20}$$

where the minus sign applies to the log-amplitude variance, and the plus sign to the phase variance. The 3D spectrum of the turbulence is indicated by $\Theta\left(0, k_\perp\right)$, where the first argument is the wavenumber component parallel to the direction of propagation, and the second argument is the wavenumber component perpendicular to it. For velocity fluctuations, the pertinent velocity component is the one parallel to the direction of propagation.

## 4.1 Scalars, Plane Waves

To determine the log-amplitude and phase variances, we need to determine the 3D spectrum $\Theta\left(0, k\right)$ from the 1D spectrum $F\left(k\right)$. A helpful intermediate step is to calculate the energy spectrum $E\left(k\right)$, which specifies the energy in the random fluctuations per unit wavenumber, from $F\left(k\right)$. For scalar fluctuations, the needed relationship is

$$E\left(k\right) = -k \frac{dF}{dk}. \tag{21}$$

Hence we have

$$E\left(k\right) = -\frac{\sigma^2 \mathcal{L} k}{\pi} \frac{dS}{dk}. \tag{22}$$

For the 3D spectrum, we multiply by 2 to go from energy to variance, and then divide by the area of a spherical shell in the wavenumber space, $4\pi k^2$. Hence

$$\Theta\left(0, k_\perp\right) = -\frac{\sigma^2 \mathcal{L}}{2\pi^2 k_\perp} \frac{dS}{dk_\perp}. \tag{23}$$

Therefore, for large wavenumbers,

$$\Theta\left(0, k_\perp\right) = \frac{p\beta\sigma^2\mathcal{L}}{2\pi^2 k_\perp^2}\left(k_\perp\mathcal{L}\right)^{-p}. \tag{24}$$

It follows from equations (20) and (23) that

$$\left\langle \chi^2, \phi^2 \right\rangle = -\frac{\Phi^2}{2}\int_0^\infty \left[1 \mp \frac{k_0}{Xk_\perp^2}\sin\frac{Xk_\perp^2}{k_0}\right]\frac{dS}{dk_\perp}\,dk_\perp. \tag{25}$$

By changing the integration variable to the normalized wavenumber $\overline{k} = k_\perp\mathcal{L}$, one can rewrite this equation as

$$\left\langle \chi^2, \phi^2 \right\rangle = -\frac{\Phi^2}{2}\int_0^\infty \left[1 \mp \frac{1}{D\overline{k}^2}\sin D\overline{k}^2\right]\frac{dS}{d\overline{k}}\,d\overline{k}\ , \tag{26}$$

in which

$$D = X/k_0\mathcal{L}^2. \tag{27}$$

By comparing the definition of $D$ to Flatté et al's diffraction parameter in equation (2), we see that the two are proportional: $D = 6\left(L/\mathcal{L}\right)^2\Lambda$. For strong diffraction ($D, \Lambda \gg 1$), contributions from the second term in the square brackets in equation (26) become negligible, and we can verify equation (3). For weak diffraction ($D, \Lambda \ll 1$), the second term is significant. In fact, assuming that most of the spectral energy is concentrated at low wavenumber (which is the case for most media of interest), the second term is approximately 1, and for the phase fluctuations we arrive at equation (4). The only nontrival case is the log-amplitude variance for weak diffraction. Here the two terms in the square brackets nearly cancel out at low wavenumber, so that the significant contribution to the integral occurs at high wavenumber. Hence it is reasonable to use the high-wavenumber limit of the spectrum, equation (10). With this substitution we find

$$\left\langle \chi^2 \right\rangle = \frac{p\beta\Phi^2}{2}\int_0^\infty \left[1 - \frac{1}{D\overline{k}^2}\sin D\overline{k}^2\right]\overline{k}^{-p-1}\,d\overline{k}. \tag{28}$$

Changing the integration variable to $u = D\overline{k}^2$, we find

$$\left\langle \chi^2 \right\rangle = \frac{p\beta}{4}D^{p/2}\Phi^2\int_0^\infty \left[1 - \frac{1}{u}\sin u\right]u^{-p/2-1}\,du. \tag{29}$$

I could not find the integral in any standard table. It can be solved, however, by first replacing the difference in square brackets with an equivalent integral representation:

$$\begin{aligned} 1-\frac{\sin u}{u} &= 1-j_0(u) \\ &= \frac{1}{2}\int_0^{\pi}\left[1-\cos(u\cos\theta)\right]\sin\theta\,d\theta \\ &= \int_0^{\pi}\sin^2\left(\frac{u\cos\theta}{2}\right)\sin\theta\,d\theta \\ &= 2\int_0^{\pi/2}\sin^2\left(\frac{u\cos\theta}{2}\right)\sin\theta\,d\theta. \end{aligned} \tag{30}$$

In the first line, $j_0$ is the spherical Bessel function. The second line follows from Poisson's integral, equation (10.1.13) in Abramowitz and Stegun (1965). The third line follows from the trigonometric identity $2\sin^2 A = 1-\cos 2A$. The fourth line is a consequence of the symmetry properties of the integrand. Substituting this result into equation (29), and reversing the order of integration between $\overline{k}$ and $\theta$, we have

$$\left\langle\chi^2\right\rangle = \frac{p\beta}{2}D^{p/2}\Phi^2\int_0^{\pi/2}\left[\int_0^{\infty}\sin^2\left(\frac{u\cos\theta}{2}\right)u^{-p/2-1}\,du\right]\sin\theta\,d\theta. \tag{31}$$

The integral within the square brackets is given by Gradshteyn and Ryzhik's (1994) equation (3.823), with the restriction $0<p<4$. We now have

$$\left\langle\chi^2\right\rangle = \frac{\beta}{2}\Gamma(1-p/2)\cos\frac{\pi p}{4}D^{p/2}\Phi^2\int_0^{\pi/2}\cos^{p/2}\theta\,\sin\theta\,d\theta, \tag{32}$$

in which $\Gamma(x)$ is the gamma function. Finally, the remaining integral is known from Gradshteyn and Ryzhik's (1994) equation (3.621.5), leading to the result

$$\left\langle\chi^2\right\rangle = \frac{\beta}{2+p}\Gamma(1-p/2)\cos\frac{\pi p}{4}D^{p/2}\Phi^2. \tag{33}$$

First we will verify that equation (33) reduces to Tatarskii's result, equation (7). We start by replacing $\Phi$, $\beta$, and $D$ with their definitions (equations (1), (12), and (27), respectively), and this leads to

$$\left\langle\chi^2\right\rangle = \frac{2\pi\gamma}{2+p}\Gamma(1-p/2)\cos\frac{\pi p}{4}k_0^{2-p/2}X^{1+p/2}. \tag{34}$$

Next, using equation (15) to eliminate $\gamma$ in favor of $C_N^2$, and performing some manipulations on the gamma functions, we have

$$\left\langle \chi^2 \right\rangle = h_s(p)\, C_N^2 k_0^{2-p/2} X^{1+p/2}, \tag{35}$$

where

$$h_s(p) = -\frac{2^{p-2}\sqrt{\pi}}{2+p}\Gamma\left(\frac{p+1}{2}\right)\frac{\cos \pi p/2}{\sin \pi p/4}. \tag{36}$$

This very general equation holds for any power-law exponent $0 < p < 4$. For turbulence ($p = 5/3$), it does reduce exactly to equation (7).

Similar manipulations on equation (33) produce

$$\left\langle \chi^2 \right\rangle = h_s(p)\, 6^{p/2} \left(\frac{L}{\mathcal{L}}\right) \Lambda^{p/2} \Phi^2. \tag{37}$$

This equation does have the basic form of equation (6). However, the factor $(L/\mathcal{L})$ is not strictly a constant. The correlation length $L$ is a property of the inertial subrange, whereas the integral length scale $\mathcal{L}$ depends on the large-scale turbulence structure. In fact, it can be shown for the atmosphere that the ratio depends strongly on the relative importance of shear and buoyancy forcings of the turbulence; for turbulence generated almost entirely by wind shear, the ratio is close to 1, whereas for turbulence generated by buoyancy, the ratio may be 1/10 or less (Wilson, 1998a). If one sets $p = 2$, and assumes that $\mathcal{L} \approx 0.2L$, as was done by Flatté et al, one finds

$$\left\langle \chi^2 \right\rangle \approx 12 \Lambda \Phi^2. \tag{38}$$

This result has the same form as equation (5), although the constant 12 appears in place of 1/4. The cause of this discrepency is unclear, since Flatté et al do not provide a derivation of their equation. The discrepency could be due, in part, to confusion in the definition of the diffraction parameter; in much of the literature it is defined with a factor of 4 in the numerator, rather than 6 in the denominator, as in equation (2).

## 4.2 Vectors, Plane Waves

The spectrum of fluctuations of an isotropic vector field differs somewhat from that of an isotropic scalar field. The energy spectrum follows from the equation (Batchelor, 1953)

$$E(k) = k^3 \frac{d}{dk}\left[\frac{1}{k}\frac{dF}{dk}\right]. \tag{39}$$

Therefore, at large wavenumber, we have

$$E(k) \simeq p(p+2)\frac{\sigma^2 \mathcal{L}\beta}{\pi}(k\mathcal{L})^{-p}. \tag{40}$$

The spectral density of the $i$th velocity component can be determined using (Batchelor, 1953)

$$\Theta\left(k_{\|}, k_{\perp}\right) = \frac{E(k)\, k_{\perp}^2}{4\pi k^4}, \tag{41}$$

where $k_{\|}$ is the wavenumber in the $i$th direction. Hence, in place of equation (24), we have

$$\Theta(0, k_{\perp}) = \frac{p(p+2)\beta\sigma^2\mathcal{L}}{4\pi^2 k_{\perp}^2}(k_{\perp}\mathcal{L})^{-p}. \tag{42}$$

This is the same as equation (24), except that it incorporates an extra factor $(p+2)/2$. Hence the scalar results apply to vectors after multiplication by this extra factor. Equivalently, one can replace $h_s(p)$ in equations (35) and (37) by

$$h_v(p) = -2^{p-3}\sqrt{\pi}\Gamma\left(\frac{p+1}{2}\right)\frac{\cos \pi p/2}{\sin \pi p/4}. \tag{43}$$

## 4.3 Scalars and Vectors, Spherical Waves

For weakly scattered spherical waves, we start with (Rytov et al 1989)

$$\left\langle \chi^2, \phi^2 \right\rangle = 2\pi^2 k_0^2 X \int_0^1 \int_0^\infty \left[1 \mp \cos\frac{X k_{\perp}^2 t(1-t)}{k_0}\right]\Theta(0, k_{\perp})\, k_{\perp}\, dk_{\perp} dt, \tag{44}$$

instead of equation (20). In the preceding, the integration over $t$ represents averaging over the propagation path. From this equation it can be shown that equations (3) and (4) are still valid for spherical waves. As with plane waves, the difficulty is in the log-amplitude variance for conditions of weak scattering and weak diffraction. Substituting equation (24) into (44), making the substituting $u = Dk^2\mathcal{L}^2$, and using the trigonometric identity $2\sin^2 A = 1 - \cos 2A$, we find that

$$\left\langle \chi^2 \right\rangle = \frac{p\beta}{2} D^{p/2}\Phi^2 \int_0^1 \int_0^\infty \left[\sin^2 \frac{ut(1-t)}{2}\right] u^{-p/2-1} du\, dt. \tag{45}$$

Performing the integrations as with the plane waves results in

$$\left\langle \chi^2 \right\rangle = \frac{\beta}{2}\Gamma\left(1-p/2\right) B\left(1+p/2, 1+p/2\right) \cos\frac{\pi p}{4} D^{p/2}\Phi^2, \tag{46}$$

where $B\left(x,y\right)$ is the beta function. Comparing this equation with equation (33), one finds that

$$\frac{\langle \chi^2 \rangle_{\text{sphere}}}{\langle \chi^2 \rangle_{\text{plane}}} = \left(1+p/2\right) B\left(1+p/2, 1+p/2\right). \tag{47}$$

This result is valid for vectors as well as scalars.

# 5. Summary

The log-amplitude variances for weak scattering and weak diffraction (the geometric acoustics regime) are summarized in tables 1 and 2. The equations in the two tables are equivalent, but the first table specifies the variances in terms of structure-function parameters, whereas the second specifies them in terms of the strength and diffraction parameters. These tables tie together and generalize the previous results of Tatarskii (1971), Flatté et al (1979), and Ostashev (1994).

An interesting conclusion of this study is that one cannot generally calculate the log-amplitude variance from the strength and diffraction parameters alone, as suggested by equation (5). Knowledge of the length-scale ratio $L/\mathcal{L}$ is also required. Alternatively, one could define a modified strength parameter based on $L$ (i.e., $\Phi_L^2 = 2\sigma^2 L k_0^2 X$) and use $\Phi_L^2$ to calculate the log-amplitude variance in the geometric acoustics regime. The distinction between the length scales would have little practical importance if the scales were nearly equal. However, in many propagation media, including the atmosphere, it appears that $\mathcal{L}$ can be an order of magnitude or more longer than $L$ (Wilson, 1998a).

Table 1. Log-amplitude variances for weak scattering and weak diffraction (the geometric acoustics regime) written in terms of structure-function parameter $C_N^2$, acoustic wavenumber $k_0$, and propagation distance $X$. The function $h_s(p)$ is defined in equation (36).

| Wave type | Fluctuations | General ($0 < p < 4$) | Turbulence ($p = 5/3$) | Internal waves ($p = 2$) |
|---|---|---|---|---|
| Plane | Scalar | $h_s(p)\, C_N^2 k_0^{2-p/2} X^{1+p/2}$ | $0.307 C_N^2 k_0^{7/6} X^{11/6}$ | $0.393 C_N^2 k_0 X^2$ |
| Plane | Vector | $\left(1+\frac{p}{2}\right) h_s(p)\, C_N^2 k_0^{2-p/2} X^{1+p/2}$ | $0.563 C_N^2 k_0^{7/6} X^{11/6}$ | $0.785 C_N^2 k_0 X^2$ |
| Spherical | Scalar | $\left(1+\frac{p}{2}\right) B\left(1+\frac{p}{2}, 1+\frac{p}{2}\right) h_s(p)\, C_N^2 k_0^{2-p/2} X^{1+p/2}$ | $0.124 C_N^2 k_0^{7/6} X^{11/6}$ | $0.131 C_N^2 k_0 X^2$ |
| Spherical | Vector | $\left(1+\frac{p}{2}\right)^2 B\left(1+\frac{p}{2}, 1+\frac{p}{2}\right) h_s(p)\, C_N^2 k_0^{2-p/2} X^{1+p/2}$ | $0.228 C_N^2 k_0^{7/6} X^{11/6}$ | $0.262 C_N^2 k_0 X^2$ |

Table 2. Log-amplitude variances for weak scattering and weak diffraction (the geometric acoustics regime) written in terms of diffraction parameter $\Lambda$, strength parameter $\Phi$, and ratio of correlation length to integral length scale, $L/\mathcal{L}$. The function $h_s(p)$ is defined in equation (36).

| Wave type | Fluctuations | General ($0 < p < 4$) | Turbulence ($p = 5/3$) | Internal waves ($p = 2$) |
|---|---|---|---|---|
| Plane | Scalar | $h_s(p)\, \frac{L}{\mathcal{L}} (6\Lambda)^{p/2} \Phi^2$ | $1.37 \frac{L}{\mathcal{L}} \Lambda^{5/6} \Phi^2$ | $2.36 \frac{L}{\mathcal{L}} \Lambda \Phi^2$ |
| Plane | Vector | $\left(1+\frac{p}{2}\right) h_s(p)\, \frac{L}{\mathcal{L}} (6\Lambda)^{p/2} \Phi^2$ | $2.51 \frac{L}{\mathcal{L}} \Lambda^{5/6} \Phi^2$ | $4.71 \frac{L}{\mathcal{L}} \Lambda \Phi^2$ |
| Spherical | Scalar | $\left(1+\frac{p}{2}\right) B\left(1+\frac{p}{2}, 1+\frac{p}{2}\right) h_s(p)\, \frac{L}{\mathcal{L}} (6\Lambda)^{p/2} \Phi^2$ | $0.553 \frac{L}{\mathcal{L}} \Lambda^{5/6} \Phi^2$ | $0.785 \frac{L}{\mathcal{L}} \Lambda \Phi^2$ |
| Spherical | Vector | $\left(1+\frac{p}{2}\right)^2 B\left(1+\frac{p}{2}, 1+\frac{p}{2}\right) h_s(p)\, \frac{L}{\mathcal{L}} (6\Lambda)^{p/2} \Phi^2$ | $1.01 \frac{L}{\mathcal{L}} \Lambda^{5/6} \Phi^2$ | $1.57 \frac{L}{\mathcal{L}} \Lambda \Phi^2$ |

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Log-amplitude and phase variances of a weakly scattered acoustic signal are calculated for line-of-sight propagation through a random medium. The spectrum of the index-of-refraction fluctuations in the random medium is assumed to scale in proportion to the wavenumber raised to an arbitrary power in the limit of large wavenumbers (small spatial scales). Both scalar and vector contributions to the index of refraction are considered. Most of the calculated results reduce to those given by Tatarskii (1971) and Ostashev (1994) when the power law exponent is –5/3, which is the value characteristic of turbulence. However, the results do not exactly reduce to an equation given by Flatte et al (1994) for the log-amplitude variance in terms of strength and diffraction parameters. The equation from Flatte et al is shown to be an approximation, strictly valid only when the spectral energy in the random medium is concentrated at a well-defined outer scale.

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